Available online at www.sciencedirect.com

SCIENCE DIRECT®

ELSEVIER

Journal of Functional Analysis 214 (2004) 247–259

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Metrics on group C^*-algebras and a non-commutative Arzelà–Ascoli theorem

Cristina Antonescu[a] and Erik Christensen[b,*]

[a] Department of Mathematics, University of California, Riverside, CA 9250 74633, USA
[b] Institut for Matematiske, Fag University of Copenhagen, Universitetsparken 5, DK 2100, Copenhagen, Denmark

Received 27 January 2003; revised 5 April 2004; accepted 5 April 2004

Communicated by Alain Connes

Abstract

On a discrete group G, a length function may implement a spectral triple on the reduced group C^*-algebra. Following A. Connes, the Dirac operator of the triple then can induce a metric on the state space of the reduced group C^*-algebra. Recent studies by M.A. Rieffel raise several questions with respect to such a metric on the state space. We propose a relaxation in the way a length function is used in the construction of a metric, and we then show that for groups of rapid decay there are many metrics related to a length function which have all the expected properties. At the end we show that this notion allows a non-commutative version of the Arzelà–Ascoli theorem.

MSC: primary 46L85; 58B34; secondary 46L07

Keywords: C^*-algebras; Discrete groups; Rapid decay; Metrics; Non-commutative topological spaces; Spectral triples

1. Introduction

In the article [Co1], Connes demonstrates that the geodesic distance on a compact, spin, Riemannian manifold $\mathcal{M}$ can be expressed in terms of an unbounded Fredholm

*Corresponding author.
E-mail addresses: cris@math.ucr.edu (C. Antonescu), echris@math.ku.dk (E. Christensen).

0022-1236/$ - see front matter
doi:10.1016/j.jfa.2004.04.015

module over the C^*-algebra $C(\mathcal{M})$. The distance between points p, q in $\mathcal{M}$ is obtained via the Dirac operator D by the formula

$$d(p,q) = \sup\{|a(p) - a(q)| \mid a \in C(\mathcal{M}), ||[D,a]|| \leqslant 1\}.$$

Inspired by the compact manifold $\mathbb{T}$, i.e. the unit circle, and the well-known identity $C^*(\mathbb{Z}) = C(\mathbb{T})$, it is natural to consider discrete groups with length functions and for such a pair (G, ℓ) to define a Dirac operator D on $l^2(G)$ by $(D\xi)(g) = \ell(g)\xi(g)$. This setup is discussed in [Co1] and Connes proves that $(l^2(G), D)$ is an unbounded Fredholm module for $C_r^*(G)$ if $\ell^{-1}([0,a])$ is a finite set for each $a \in \mathbb{R}_+$. Still following [Co1], one can then try to define a metric on the state space $\mathcal{S}(C_r^*(G))$ of $C_r^*(G)$ by

$$d_\ell(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| \mid a \in C_r^*(G), ||[D,a]|| \leqslant 1\}.$$

In the first place it is not clear that $d_\ell(\varphi,\psi) < \infty$ for all pairs, but if that is the case, then one could ask if $\mathcal{S}(C_r^*(G))$ is a bounded set with respect to this metric, and if so does d_ℓ generate the w*-topology on $\mathcal{S}(C_r^*(G))$.

Especially Rieffel [Ri1,Ri2,Ri3,Ri4] has studied these questions and some more general questions concerning the set of all metrics on $\mathcal{S}(C_r^*(G))$, where G is a group with reasonable properties. We were inspired by Rieffels works and tried to answer some of the questions he raised in the context of free non-Abelian groups. Based on the techniques introduced by Haagerup in [Haa], we could prove that for the reduced group C^*-algebras of such groups the metric is bounded. In the article [OR] by Ozawa and Rieffel they obtained much better results and even for a larger class of discrete groups, namely the groups which satisfy the Haagerup property [OR]. As a consequence we do not include our results concerning free groups in this paper, which consequently is shorter than the preprint version. Since the title of the preprint refers to some of the results we have cut out, we have changed the title too.

During the work on the free group case we realized that for discrete groups, which have the property, named *rapid decay*, it is quite easy to construct metrics on the state space of the reduced C^*-algebra such that these metrics have all the properties asked for. The concept *groups of rapid decay* was studied by Jollisaint [Jo] and it will be defined properly in Section 2. For a discrete group, which is of rapid decay with respect to some length function ℓ there exists a $k \in \mathbb{N}$ such that the metric d_ℓ^k on the state space $\mathcal{S}$ of $C_r^*(G)$ defined by

$$d_\ell^k(\varphi,\psi) = \sup\{|\varphi(a) - \psi(a)| \mid a \in C_r^*(G), ||\underbrace{[D,[D,\ldots,[D,a]\ldots]]}_{k}|| \leqslant 1\}.$$

has all the wanted properties. The proof of this result is on the other hand very easy, and this may be seen as an indication in favor of this alternative construction of a metric from a length function. This alternative way of obtaining a metric from *higher derivatives* made it clear to us that metric spaces without a smooth structure might be of interest in the non-commutative world too, so we have tried to look for properties of non-commutative metric spaces. We suggest a definition and show that it works well with respect to a translation of the classical Arzelà–Ascoli theorem into a non-commutative language. It turns out that a metric on a compact space, say X, induces a compact convex balanced subset, say $\mathcal{K}$, of $C(X)$ which separates the points in X. The Arzelà–Ascoli theorem may then be rephrased using the bipolar theorem, such that for a subset, say $\mathcal{H}$, of $C(X)$ it turns out that $\mathcal{H}$ is relatively compact if and only if

$$\mathcal{H} \quad \text{is bounded and} \quad \forall \varepsilon > 0 \ \exists N \in \mathbb{R}_+, \quad \mathcal{H} \subseteq \mathcal{A}_\varepsilon + N\mathcal{K} + \mathbb{C}I.$$

Here $\mathcal{A}_\varepsilon$ denotes the closed ball in $\mathcal{A}$ of radius ε.

This result emphasizes that the concept of a metric on the state space $\mathcal{S}(\mathcal{A})$ generating the w*-topology is closely related to compact, convex and state separating subsets of $\mathcal{A}$. Once you know one such set, then you can decide on the relative compactness of any other subset of $\mathcal{A}$. This close connection between metrics on $\mathcal{S}(\mathcal{A})$ and relatively compact subsets of $\mathcal{A}$ is studied in details in the papers [Ri2,Pav]; from where we have learned it. We would like to thank G. Pisier, M.A. Rieffel and N. Ozawa for some most helpful comments and for being very open minded.

2. Non-commutative metric spaces and discrete groups of rapid decay

We will study properties of the C^*-algebra generated by the left regular representation λ of a discrete group G on $l^2(G)$. We refer to Chapter 6 of [KR] for the basic properties of this C^*-algebra, but we will use a slightly different notation which is inspired by Connes' presentations in [Co1,Co2]. This means that for an x in the group algebra $\mathbb{C}G$ we will write $\lambda(x) = \sum_g x(g)\lambda_g$ for the convolution operator on $l^2(G)$, and for $g \in G$ we let δ_g denote the natural basis element in $l^2(G)$. The C^*-algebra generated by $\lambda(\mathbb{C}G)$ in $B(l^2(G))$ is called the reduced group C^*-algebra and denoted $\mathrm{C}_r^*(G)$. Any element x in this algebra has a unique representation in $l^2(G)$ by $x \to x\delta_e$ so in a natural way we have

$$l^1(G) \subseteq C_r^*(G) \subseteq l^2(G)$$

and for $x \in l^1(G)$

$$||x||_2 \leqslant ||\lambda(x)|| \leqslant ||x||_1.$$

For a discrete group of rapid decay (RD) one has a kind of inverse to the first inequality and such a type of inequality is very powerful as we shall see. In order to explain the concept of rapid decay we remind the reader that a length function ℓ on a group G is a mapping $\ell : G \to \mathbb{R}_+ \cup \{0\}$ such that

(i) $\ell(e) = 0$,
(ii) $\forall g, h \in G : \ell(gh) \leqslant \ell(g) + \ell(h)$,
(iii) $\forall g \in G : \ell(g) = \ell(g^{-1})$.

Definition 2.1. A discrete group G is said to be of rapid decay (RD) with constant C and eksponent s if there exist a length function ℓ on G and positive reals C, s such that

$$\forall x \in \mathbb{C}G : ||\lambda(x)|| \leqslant C\left(\sum_{g \in G} (1+\ell(g))^{2s}|x(g)|^2\right)^{\frac{1}{2}}.$$

In the very innovative paper [Haa], Haagerup proved that the free non-Abelian groups $\mathbb{F}_n$ are of rapid decay with $C = 2$ and $s = 2$. Also the free Abelian groups $\mathbb{Z}^k$ are of rapid decay and in fact one can get an estimate dominating the norm $||x||_1$. For $k = 1$ this is obvious since for $x \in \mathbb{C}\mathbb{Z}$

$$\sum_{n \in \mathbb{Z}} |x_n| \leqslant \left(\sum_{n \in \mathbb{Z}} (1+|n|)^2|x_n|^2\right)^{\frac{1}{2}}\left(\sum_{n \in \mathbb{Z}} (1+|n|)^{-2}\right)^{\frac{1}{2}}.$$

The article [Jo] by Jolissaint contains a lot of results on groups of rapid decay and among his results he proves that a discrete group is of rapid decay, if it is of polynomial growth with respect to some set of generators and the corresponding length function. In Connes' book [Co2] he presents in Theorem 5, p. 241 a proof of the fact that the word hyperbolic groups of Gromov all are of rapid decay.

We will return to discrete groups of rapid decay shortly, but first we will study higher commutators by Dirac operators in the general setting of a discrete group with a length function satisfying a finiteness condition called proper, which is defined below.

Definition 2.2. Let G be a discrete group and $\ell : G \to \mathbb{R}_+ \cup \{0\}$ a length function. If, for each $c \in \mathbb{R}_+$, $\ell^{-1}([0, c])$ is a finite set then we say that ℓ is proper.

As mentioned before Connes defines in [Co1] a metric on a non-commutative C^*-algebra via an unbounded Fredholm module. For a discrete group G with a proper length function ℓ he then constructs an unbounded Fredholm module.

This unbounded Fredholm module, over $C^*_r(G)$, where G is a discrete group with a proper length function, is the Hilbert space $l^2(G)$ combined with the Dirac operator D on $l^2(G)$, which is the selfadjoint unbounded multiplication operator, which multiplies $\xi \in l^2(G)$ by ℓ pointwise.

In [Ri2], Rieffel introduces the concept of an unbounded lower semicontinuous Lipschitz seminorm L on a C^*-algebra $\mathcal{A}$. The seminorm is usually not bounded nor everywhere defined since it is a generalization of the kind of seminorm you would get on the continuous functions on a manifold by taking the uniform norm on some derivative of a differentiable function. The term *Lipschitz* means that the kernel of the seminorm consists of the scalars and the term *lower semicontinuous* means that the set $\{a \in \mathcal{A} \mid L(a) \leqslant 1\}$ is norm closed. In our context the operator D induces several lower semicontinuous Lipschitz seminorms whose domains of definition always contain the dense subalgebra $\lambda(\mathbb{C}G)$ of $C^*_r(G)$.

In order to define these seminorms we fix the setting as above. Let G be a discrete group with a proper length function $\ell : G \to \mathbb{R}_+ \cup \{0\}$ such that $\ell^{-1}(0) = \{e\}$. Let D denote the corresponding Dirac operator, then for certain elements of $C_r^*(G)$ we want to study the higher commutators $[D, [D, \dots, [D, \lambda(a)] \dots]]$. We will first show that for $a \in \mathbb{C}G$ they are all bounded even though they are not everywhere defined. We think that the most easy way to get a picture of these commutators is by looking at operators on $H = l^2(G)$ as infinite matrices with respect to the orthonormal basis $\{\delta_g : g \in G\}$. In this setup D becomes a closed-positive diagonal operator with diagonal elements given by $D_{gg} = \ell(g)$. Then it is quite easy to see that we can make a formal commutator between the diagonal matrix (D_{gg}) and any infinite matrix (a_{gh}) and we will get

$$[(D_{gg}), (a_{gh})]_{st} = (\ell(s) - \ell(t))a_{st}$$

and for k consecutive commutation operations we will have the matrix

$$[(D_{gg}), [, \dots, (a_{gh})] \dots]_{st} = (\ell(s) - \ell(t))^k a_{st}.$$

For the unitaries λ_g, we will now show that the norm of such a higher commutator is bounded by the number $\ell(g)^k$. So let $\xi = \sum \xi(g)\delta_g$ be in $l^2(G)$ then for k consecutive commutation operations and any $f \in G$ we get

$$[D, [, \dots [D, \lambda_g] \dots]]\xi(f) = (\ell(f) - \ell(g^{-1}f))^k \xi(g^{-1}f).$$

By the triangle inequality for length functions the claim then follows, and we can conclude that for any k the higher commutator between D and any element $a \in \mathbb{C}G$ is a bounded operator. We will denote this operator $\Delta^k(a)$ and define Δ^k as a densely defined unbounded operator from $C_r^*(G)$ to $B(H)$ by

$$\mathrm{dom}(\Delta^k) = \{a \in C_r^*(G) : \text{the matrix } ((\ell(s) - \ell(t))^k a_{st}) \text{ is bounded}\}$$

$$\text{and for } a \in \mathrm{dom}(\Delta^k) : \Delta^k(a) \text{ is the bounded operator}$$

$$\text{with matrix } ((\ell(s) - \ell(t))^k a_{st}).$$

From the definition of the operators Δ^k it follows easily, as we shall see, that they are all closed. Suppose that for a $k \in \mathbb{N}$ and a null sequence (a_n) from $\mathrm{dom}(\Delta^k)$ the sequence $(\Delta^k(\lambda(a_n)))$ converges towards $b \in B(H)$. Then by the known action of Δ^k it follows that the matrix of b will consist entirely of zeros, and the operator Δ^k is closed.

After this general description of some properties of the operators Δ^k, we can describe the seminorms L_D^k we are going to use by.

Definition 2.3. For any $k \in \mathbb{N}$ L_D^k is defined by: $\mathrm{dom}(L_D^k) = \mathrm{dom}(\Delta_D^k)$ and for any a in this domain $L_D^k(a) := ||\Delta^k(\lambda(a))||$.

We may then show, that these seminorms behave nicely.

Proposition 2.4. *Let G be a discrete group with a proper length function ℓ such that $\ell^{-1}(0) = \{e\}$. Then for any natural number k the seminorm L_D^k on $C_r^*(G)$ is Lipschitz and lower semicontinuous.*

Proof. Suppose $a \in \mathrm{dom}(L_D^k)$ satisfies $L_D^k(a) = 0$. Since the unit vector δ_e is in the kernel of D we get

$$0 = L_D^k(a) = ||\Delta^k(\lambda(a))|| \geqslant ||\Delta^k(\lambda(a))\delta_e|| = \left|\left|\sum_{g \in G} \ell(g)^k a(g)\delta_g\right|\right|.$$

Hence, for all $g \neq e$ we have $a(g) = 0$ so $a = a(e)I$ and L_D^k is a Lipschitz seminorm.

In order to see that it is lower semicontinuous we suppose that (a_n) is a convergent sequence from $\mathrm{dom}(L_D^k)$ with limit, say b in the C^*-algebra such that for each $n \in \mathbb{N}$ we have $L_D^k(a_n) \leqslant 1$. When looking at the infinite matrices, it is easy to see that the formal matrix $((\ell(s) - \ell(t))^k b_{st})$ has the property that any finite submatrix is of norm at most 1. Hence, we see that b is in dom L_D^k and $L_D^k(b) \leqslant 1$, so the proposition follows. $\square$

Having this situation, we can define some possibly unbounded metrics on the state space of the reduced group C^*-algebra of G.

Definition 2.5. Let G be a discrete group with a proper length function ℓ, let k be a natural number and let $\mathcal{S}$ denote the state space of $C_r^*(G)$ then $d_\ell^k : \mathcal{S} \times \mathcal{S} \to [0, \infty]$ is defined by

$$d_\ell^k(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| \ : \ a \in \mathrm{dom}(L_D^k) \quad \text{and} \quad L_D^k(a) \leqslant 1\}.$$

Since for any group element g we have $L_d^k(\ell(g)^{-k}\lambda_g) \leqslant 1$ we see that d_ℓ^k must separate the points in $\mathcal{S}$. It is not clear if $d_\ell^k(\varphi, \psi) < \infty$ always, but except for that, d_ℓ^k behaves exactly as a metric on $\mathcal{S}$ so we will allow this slight abuse of language and call d_ℓ^k a metric on $\mathcal{S}$.

Before we state and prove our result we want to mention that it follows from Rieffel's works and the work of Pavlović that a Lipschitz seminorm L on a unital C^*-algebra $\mathcal{A}$ is bounded and induces the w*-topology on the state space of $\mathcal{A}$ if and only if the set

$$\{a \in \mathcal{A} : L(a) \leqslant 1\}$$

has a relatively compact image in the quotient space $\mathcal{A}/\mathbb{C}I$, equipped with the quotient norm. This result is the basis for our investigations in Section 3, and a proof of it is included in Lemma 3.1.

Theorem 2.6. *Let G be a discrete group with a proper length function $\ell : G \to \mathbb{R}_+ \cup \{0\}$ such that $\ell^{-1}(e) = \{0\}$, and G is of rapid decay with respect to ℓ. Then there exists a $k_0 \in \mathbb{N}$ such that for any natural number $k \geqslant k_0$ the metric generated by L_D^k on $\mathcal{S}(C_r^*(G))$ is bounded and the topology generated by the metric equals the w^*-topology.*

Proof. We know that L_D^k is a lower semicontinuous Lipschitz seminorm, so when we have proven prove that the set

$$\tilde{\mathcal{K}}_k = \{a \in C_r^*(G) : L_D^k(a) \leqslant 1\}$$

has relatively compact image in the quotient space $\mathcal{A}/\mathbb{C}I$ we will actually get that this set is compact, since the quotient mapping is an open mapping. We will now turn to the proof of the relative compactness of $\tilde{\mathcal{K}}_k/\mathbb{C}I$. Instead of working in the quotient space we prefer to work on a representative of the quotient space, namely the cross-section $\mathcal{K}_k$ which is the subset of $\tilde{\mathcal{K}}_k$ consisting of those elements of trace zero.

$$\mathcal{K}_k = \{a \in C_r^*(G) : L_D^k(a) \leqslant 1 \quad \text{and} \quad (a\delta_e, \delta_e) = 0\}.$$

The assumption of rapid decay implies that there exist two positive reals C, s such that

$$\forall x \in \mathbb{C}G \ ||\lambda(x)|| \leqslant C\left(\sum_g (1 + \ell(g))^{2s} |\lambda(g)|^2\right)^{\frac{1}{2}}$$

The number k_0 is then defined by $k_0 = \lfloor s \rfloor + 1$, and given this we will fix a $k \in \mathbb{N}$ such that $k \geqslant k_0$. The first observation we need has already been used before, namely that any element $a \in C_r^*(G)$ can be expressed as an l^2 convergent infinite sum $\sum a(g)\delta_g$ and that $||a||_2 = ||\lambda(a)\delta_e||$. Having this, and the fact that $D\delta_e = 0$ we get for an $a \in \mathcal{K}_k$ that

$$1 \geqslant L_D^k(a) = ||\Delta^k(\lambda(a))|| \geqslant ||\Delta^k(\lambda(a))\delta_e|| = \left|\left|\sum \ell(g)^k a(g)\delta_g\right|\right|.$$

In particular, we get for an $a \in \mathcal{K}_k$ that

$$\sum \ell(g)^{2k} |a(g)|^2 \leqslant 1.$$

The properness condition on $\ell(g)$ implies that there are only finitely many group elements of length less than any natural number n. Hence, in order to prove that $\mathcal{K}_k$ is relatively compact it is sufficient to show that for any positive real ε there exists a real number $t \geqslant 1$ such that for any $a \in \mathcal{K}_k$

$$\left|\left|\sum_{\ell(g) \geqslant t} a(g)\lambda_g\right|\right|_{C_r^*(G)} \leqslant \varepsilon$$

but this is on the other hand easily obtainable from the inequality at the top of the proof. In fact let $t \in \mathbb{R}, t \geqslant 1$, then for $g \in G$ with $\ell(g) \geqslant t \geqslant 1$ we get

$$(1+\ell(g))^{2s} \leqslant 2^{2s}\ell(g)^{2s} \leqslant 2^{2s}t^{(2s-2k)}\ell(g)^{2k}.$$

Since $2s-2k<0$ there exists a $t \in \mathbb{R}_+, t \geqslant 1$ such that $2^{2s}t^{(2s-2k)} \leqslant \frac{\varepsilon^2}{C^2}$. For this t we then obtain

$$\left\|\sum_{\ell(g)\geqslant t} a(g)\lambda_g\right\|^2_{C_r^*(G)} \leqslant C^2 \sum_{\ell(g)\geqslant t} (1+\ell(g))^{2s}|a(g)|^2 \leqslant C^2 \frac{\varepsilon^2}{C^2} L_D^k(a)^2 \leqslant \varepsilon^2$$

Hence $\mathcal{K}_k$ is relatively compact in $\mathcal{A}$, and the theorem follows. $\square$

3. A non-commutative Arzelà–Ascoli theorem

Given the common use of language which says: *a non-commutative compact topological space is a unital C^*-algebra*, it seems natural to propose a definition of a metric on this non-commutative space in terms of an object which is related directly to the algebra. If there is an obvious smooth structure in terms of a spectral triple, this object should be preferred since it contains much more information, but for a general non-commutative, unital and separable C^*-algebra $\mathcal{A}$ without any particularities, it seems that any norm compact balanced and convex subset of $\mathcal{A}$ which separates the states on $\mathcal{A}$ contains all the information needed.

Definition 3.1. Let $\mathcal{A}$ be a unital C^*-algebra. A subset $\mathcal{K}$ of $\mathcal{A}$ is called a metric set if it is norm compact, balanced, convex and separates the states on $\mathcal{A}$.

This point of view is nearly the same as the one Rieffel has in mind [Ri2] in his studies of Lipshitz seminorms. The two concepts are connected in the following way. Given a metric set, say $\mathcal{K}$, then the Minkowski functional associated to the closed convex set $\mathcal{K} + \mathbb{C}I$ is a lower semicontinuous Lipschitz seminorm. The reason why we propose to look at such a set as a non-commutative metric is partly due to our reading of the works by Rieffel and Pavlović, but also partly due to the translation of the classical Arzelà–Ascoli theorem into the non-commutative language which we present below.

Following the ideas of Rieffel [Ri1] we deduce the following result which justifies the definition of a metric set.

Proposition 3.2. *Let $\mathcal{A}$ be a unital C^*-algebra, $\mathcal{S}$ the state space of $\mathcal{A}$ and $\mathcal{K}$ a norm compact subset of $\mathcal{A}$ which separates the points in the state space. Then for states φ, ψ*

on $\mathcal{A}$ the formula

$$d_{\mathcal{K}}(\varphi, \psi) \coloneqq \sup_{k \in \mathcal{K}} |(\varphi - \psi)(k)|$$

defines a metric on the state space $\mathcal{S}$ which generates the w^*-topology.

Proof. The separation property and the compactness assumption show that $d_{\mathcal{K}}$ is a bounded metric on $\mathcal{S}$.

The norm compactness of $\mathcal{K}$ and the boundedness of $\mathcal{S}$ further implies that the topology induced by $d_{\mathcal{K}}$ is a Hausdorff topology weaker than the compact w*-topology on $\mathcal{S}$.

A well-known theorem from topology then tells that the two topologies do agree, and the proposition follows. $\square$

In the proposition, just proven, the set $\mathcal{K}$ is just a norm compact set which separates the states, whereas a metric set has to be convex and balanced too. The reason why we have made this choice in the definition stems from the fact that the metric we obtained from $\mathcal{K}$ is the same as the one we would have obtained if $\mathcal{K}$ was replaced by it's closed balanced convex hull.

With this definition at hand one can easily construct metric sets for separable unital C^*-algebras and for instance for a countable group $G = \{g_n \mid n \in \mathbb{N}\}$ a metric set in $C_r^*(G)$ could be given by the following expression where $\overline{\text{conv}}$ means the closed convex hull.

$$\mathcal{K} \coloneqq \overline{\text{conv}}\left(\bigcup_{n=1}^{\infty} \{\alpha\lambda_{g_n} + \beta\lambda_{g_n}^* : \alpha, \beta \in \mathbb{C} \quad \text{and} \quad |\alpha| + |\beta| \leqslant 1/n\}\right).$$

The classical Arzelà–Ascoli theorem gives a characterization of relatively compact subsets of $C(X)$ for a compact topological space X. If X is equipped with a metric ρ generating the topology on X one can construct a convex subset $\tilde{\mathcal{K}}$ of $C(X)$ by

$$\tilde{\mathcal{K}} = \{f \in C(X) : \forall x, y \in X \; |f(x) - f(y)| \leqslant \rho(x, y)\}$$

This set is unbounded since any constant function belongs to $\tilde{\mathcal{K}}$. If one normalizes the set by considering the subset consisting of those elements which all vanish at a certain point x_0 then the Arzelà–Ascoli theorem immediately shows that the set, say $\mathcal{K}$, given by

$$\mathcal{K} = \{f \in C(X) : \forall x, y \in X \; |f(x) - f(y)| \leqslant \rho(x, y) \quad \text{and} \quad f(x_0) = 0\}$$

will be a compact balanced convex subset of $C(X)$. To see that $\mathcal{K}$ also separates the states on $C(X)$ is a bit complicated, so we will show that $\mathcal{K}$ separates the points in X first and then give an indication of how a general separation result for states might be. Suppose y, z are points in X then the function f given by $f(x) = \rho(x, z) - \rho(x_0, z)$ is in $\mathcal{K}$ and it follows that $\rho(y, z) = f(y) - f(z)$, so it is possible to recover ρ from $\mathcal{K}$

by the usual formula. By the Monge–Kantorovich construction as described for instance in [Ri2], $\mathcal{K}$ induces also a metric, say $\tilde{\rho}$, on the Borel probability measures, i.e. the states on $C(X)$, such that $\tilde{\rho}$ is an extension of the given metric ρ on X. In particular we see that $\mathcal{K}$ separates the states on $C(X)$.

A way to look at the Arzelà–Ascoli theorem is to see that the well-known equicontinuity condition on a subset $\mathcal{H}$ of $C(X)$ is just a way of comparing $\mathcal{H}$ to a multiple of the bipolar of $\mathcal{K}$. What we do in the following lines is just to transfer this measuring process to the non-commutative case, and in the meantime the not so precise statements just above will, hopefully, be understandable. The methods we use are elementary functional analytic duality results. So we have wondered if this sort of result is valid in a much wider generality like operator spaces [ER,Ke]. It seems that the validity of a generalization of Lemma 3.3 to this new setting is the crucial thing. Before we start we want to introduce some more notation. We will be considering the self-adjoint part of a unital C^*-algebra which is denoted $\mathcal{A}_h$ and we want to think of the elements in $\mathcal{A}$ as affine complex w*-continuous functions on the state space $\mathcal{S}$ of $\mathcal{A}$, so we will let $\mathbb{A}(\mathcal{S})$ denote the space of w*-continuous affine complex functions on $\mathcal{S}$ and for an element $a \in \mathcal{A}$, $\widehat{a}$ will denote the corresponding affine function in $\mathbb{A}(\mathcal{S})$. This representation of $\mathcal{A}$ is called Kadison's functional representation of $\mathcal{A}$. It is well known that the functional representation is isometric on $\mathcal{A}_h$, but for a general element $a \in \mathcal{A}$ we only have the estimates

$$||a|| \geqslant \sup|\widehat{a}(\varphi)| = ||\widehat{a}|| \geqslant \tfrac{1}{2}||a||.$$

This shows that a subset $\mathcal{H}$ of $\mathcal{A}$ is bounded if and only if the subset $\widehat{\mathcal{H}}$ of $\mathbb{A}(\mathcal{S})$ is bounded. The term balanced is used with respect to the field of complex numbers unless we explicitely look at subsets consisting of self adjoint operators, only. This means that a subset $\mathcal{H}$ of $\mathcal{A}$ is balanced if for any complex number μ such that $|\mu| \leqslant 1$ we have $\mu\mathcal{H} \subseteq \mathcal{H}$. We remind the reader that a subscript attached to a Banach space like Y_ε means that we consider the closed ball of radius ε in Y and an asterix attached to Y like Y^* means the dual space. For a pair of Banach spaces like $\mathcal{A}$ and $\mathcal{A}^*$ we will use the duality result known under the name of the bipolar theorem. Here, the polar of a set $\mathcal{H} \subseteq \mathcal{A}$ is denoted $\mathcal{H}^\circ$ and defined by

$$\mathcal{H}^\circ = \{\gamma \in \mathcal{A}^* \mid \forall h \in \mathcal{H} \mid \gamma(h)| \leqslant 1\}$$

The bipolar theorem with respect to this polar, then states that the bipolar $\mathcal{H}^{\circ\circ}$, which now is a subset of $\mathcal{A}$, is the smallest balanced, convex and norm closed set in $\mathcal{A}$ which contains $\mathcal{H}$.

We can now start the presentation of the generalization of the Arzelà–Ascoli theorem and our first lemma is closely connected to the very fundamental structure in C^*-algebra theory, that a continuous real linear functional can be decomposed in a unique way into a difference of two-positive functionals, such that a certain norm identity holds.

Lemma 3.3. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{S}$ the state space of $\mathcal{A}$, then*

$$\mathcal{S} - \mathcal{S} = (\mathcal{A}_h^*)_2 \cap \{\mathbb{C}I\}^{\perp}.$$

Proof. The inclusion "$\subseteq$" is obvious. To prove the remaining inclusion "$\supseteq$" let us take an arbitrary element f in $(\mathcal{A}_h^*)_2 \cap \{\mathbb{C}I\}^{\perp}$. It is well known that for f in $(\mathcal{A}_h^*)_2$ we can find two positive linear functionals f^+ and f^- such that $f = f^+ - f^-$ and $||f|| = ||f^+|| + ||f^-||$. If $f = 0$ we can write f as a difference $g - g$ where g is any state on the unital algebra $\mathcal{A}$. If $f \neq 0$ the condition $f(I) = 0$ implies that $0 \neq ||f^+|| = ||f^-|| = \frac{1}{2}||f|| \leqslant 1$. Based on f^+ we can then define a positive functional g of norm $||g|| = 1 - ||f^+||$ by

$$g = \frac{(1 - ||f^+||)}{||f^+||} f^+.$$

By construction it follows that $f^+ + g$ and $f^- + g$ are both states and from the equality

$$f = (f^+ + g) - (f^- + g)$$

we can conclude that $f \in \mathcal{S} - \mathcal{S}$, and the lemma follows. □

We can now state and prove the result of this section.

Theorem 3.4. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{K}$ a metric subset of $\mathcal{A}$. For any subset $\mathcal{H}$ of $\mathcal{A}$ the following conditions are equivalent*:

(i) *The set $\mathcal{H}$ is norm relatively compact.*
(ii) *The set of affine functions $\{\widehat{h} \in \mathbb{A}(\mathcal{S}) \mid h \in \mathcal{H}\}$ is bounded and equicontinuous with respect to the w^*-topology on $\mathcal{S}$.*
(iii) *The set $\mathcal{H}$ is bounded and for every $\varepsilon > 0$ there exists a real $N > 0$ such that*

$$\mathcal{H} \subseteq \mathcal{A}_\varepsilon + N\mathcal{K} + \mathbb{C}I.$$

Proof. The equivalence between (i) and (ii) follows from the classical Arzelà–Ascoli theorem and the fact mentioned above that $\mathcal{H}$ is bounded if and only if $\widehat{\mathcal{H}}$ is bounded. To prove the equivalence between (ii) and (iii) we start with (iii) $\Rightarrow$ (ii). From the boundedness of $\mathcal{H}$ it follows that the set $\{\widehat{h} \mid h \in \mathcal{H}\}$ is bounded. To prove the equicontinuity of this set let us fix an $\varepsilon > 0$ and find a positive real N which fulfills the condition (iii) with respect to $\frac{\varepsilon}{4}$. Moreover, let φ and ψ be two states such that

$$d_{\mathcal{K}}(\varphi, \psi) < \frac{\varepsilon}{2N}.$$

then we will show that for any $h \in \mathcal{H}$, $|\widehat{h}(\varphi) - \widehat{h}(\psi)| \leq \varepsilon$. Let now h be an arbitrary element in $\mathcal{H}$, then by (iii) we can find an element $a \in \mathcal{A}_1$, an element $k \in \mathcal{K}$ and a complex number μ such that

$$h = \frac{\varepsilon}{4}a + Nk + \mu 1.$$

then we obtain

$$|\widehat{h}(\varphi) - \widehat{h}(\psi)| = |(\varphi - \psi)(h)| \tag{3.1}$$

$$\leqslant \left|(\varphi - \psi)\left(\frac{\varepsilon}{4}a\right)\right| + |(\varphi - \psi)(Nk)| \tag{3.2}$$

$$\leqslant \frac{\varepsilon}{2} + Nd_{\mathcal{K}}(\varphi, \psi) \tag{3.3}$$

$$< \varepsilon. \tag{3.4}$$

and the equicontinuity of $\widehat{\mathcal{H}}$ has been established.

In order to prove the last implication (ii) $\Rightarrow$ (iii) we again first mention that $\mathcal{H}$ is bounded. Let then $\varepsilon > 0$ be given and find, via the equicontinuity assumption on $\widehat{\mathcal{H}}$, a $\delta > 0$ such that

$$\forall h \in \mathcal{H} \forall \varphi, \psi \in \mathcal{S}: \quad d_{\mathcal{K}}(\varphi, \psi) \leqslant \delta \Rightarrow |(\varphi - \psi)(h)| = |\widehat{h}(\varphi) - \widehat{h}(\psi)| \leqslant \varepsilon.$$

We will now use the bipolar theorem and remark that the expression $d_{\mathcal{K}}(\varphi, \psi) \leqslant \delta$ exactly means that $\varphi - \psi \in \delta(K^\circ)$. It is clear that $\varphi - \psi \in \mathcal{S} - \mathcal{S}$ and an application of Lemma 3.3 then shows that the implication above can just as well be expressed as

$$\forall h \in \mathcal{H} \forall \gamma \in (\mathcal{A}_h^*)_2 \cap \{\mathbb{C}I\}^\perp \cap \delta(\mathcal{K}^\circ) : |\gamma(h)| \leqslant \varepsilon.$$

This statement is not sufficient for our computations because it involves the space $\mathcal{A}_h^*$ rather that just $\mathcal{A}^*$. Since a functional on $\mathcal{A}$ vanishes on the identity I if and only both it's hermitian and it's skew hermitian part vanish on I, we can change from $\mathcal{A}_h^*$ to $\mathcal{A}^*$ at the compensation of a factor of 2, so we have

$$\forall h \in \mathcal{H} \forall \gamma \in \mathcal{A}_2^* \cap \{\mathbb{C}I\}^\perp \cap \delta(\mathcal{K}^\circ) : |\gamma(h)| \leqslant 2\varepsilon.$$

Since all the sets involved now are convex and balanced the bipolar theorem can be applied very easily. Moreover, $\mathcal{K}$ is norm compact so any set of the form $\mathcal{A}_\varepsilon + \mathbb{C}I + N\mathcal{K}$ is norm closed balanced and convex. The relation we just established gives immediately the first inclusion below and the rest follows by some well-known "polar techniques" and an application of the bipolar theorem.

$$\mathcal{H} \subseteq 2\varepsilon(\mathcal{A}_2^* \cap \{\mathbb{C}I\}^\perp \cap \delta(\mathcal{K}^\circ))^\circ \tag{3.5}$$

$$= 2\varepsilon\left(\mathcal{A}_{\frac{1}{2}} \cup \mathbb{C}I \cup \frac{1}{\delta}\mathcal{K}\right)^{\circ\circ} \tag{3.6}$$

$$= 2\varepsilon\overline{\text{conv}}\left(\mathcal{A}_{\frac{1}{2}} \cup \mathbb{C}I \cup \frac{1}{\delta}\mathcal{K}\right) \tag{3.7}$$

$$\subseteq 2\varepsilon\left(\mathcal{A}_{\frac{1}{2}} + \mathbb{C}I + \frac{1}{\delta}\mathcal{K}\right). \tag{3.8}$$

In conclusion for the given ε we found the number $N = \lceil \frac{2\varepsilon}{\delta} \rceil$ such that

$$\mathcal{H} \subseteq \mathcal{A}_\varepsilon + \mathbb{C}I + N\mathcal{K}$$

which proves the desired implication. □

References

[Co1] A. Connes, Compact metric spaces, Fredholm modules, and hyperfiniteness, Ergodic Theory Dynamical Systems 9 (1989) 207–220.

[Co2] A. Connes, Noncommutative Geometry, Academic Press, San Diego, 1994.

[ER] E.G. Effros, Z.-J. Ruan, Operator Spaces, Oxford University Press, Oxford, 2000.

[Haa] U. Haagerup, An example of a non-nuclear C^*-algebra which has the metric approximation property, Invent. Math. 50 (1979) 279–293.

[Jo] P. Jolissaint, Rapidly decreasing functions in reduced C^*-algebras of groups, Trans. Amer. Math. Soc. 317 (1990) 167–196.

[KR] R.V. Kadison, J.R. Ringrose, Fundamentals of the Theory of Operator Algebras, Academic Press, New York, 1983.

[Ke] D. Kerr, Matricial quantum Gromov–Hausdorff distance, arXiv:math.OA/0207282 v1, 2002.

[OR] N. Ozawa, M.A. Rieffel, Hyperbolic group C^*-algebras and free-product C^*-algebras as compact quantum metric spaces, arXiv:math.OA/0302310.

[Pav] B. Pavlović, Defining metric spaces via operators from unital C^*-algebras, Pacific J. Math. 186 (1998) 285–313.

[Ri1] M.A. Rieffel, Metrics on states from actions of compact groups, Doc. Math. 3 (1998) 215–229.

[Ri2] M.A. Rieffel, Metrics on state spaces, Doc. Math. 4 (1999) 559–600.

[Ri3] M.A. Rieffel, Gromov–Hausdorff distance for quantum metric spaces, arXiv: math. OA/0011063, Mem. Amer. Math. Soc. 168 (796) (2004) 1–65.

[Ri4] M.A. Rieffel, Group C^*-algebras as compact quantum metric spaces, Doc. Math. 7 (2002) 605–651.